Localization of classical waves in two-dimensional random media: A comparison between the analytic theory and exact numerical simulation

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The localization length of classical waves in two-dimensional random media is calculated exactly, and is compared with the theoretical prediction from the current analytic theory. Significant discrepancies are observed. It is also shown that as the frequency varies, critical changes in the localization behavior can occur. However, by a rescaling of parameters the two results tend to match each other for weak scattering. Possible reasons for the discrepancies are discussed.

I. INTRODUCTION

Propagation of waves through random medium has been and continues to be a subject of vivid research [1–6]. When propagating through media containing many scatterers, waves will be repeatedly scattered by each scatterer, forming a multiple scattering process [7]. Multiple scattering is responsible for phenomena such as random laser [8,9], electronic transport in impure solids [10], and photonic or acoustic band gaps [11–13]. Under proper conditions, multiple scattering leads to the unusual phenomenon of wave localization, a concept introduced by Anderson [10].

Such a localization phenomenon has been characterized by two levels. One is the weak localization, associated with the enhanced backscattering due to constructive interference from the reversed propagating paths. The second is termed as the strong localization (for brevity just termed as localization hereafter), in which significant inhibition of transmission surfaces, indicating that the energy mostly remains in a region of space in the neighborhood of the emission. The generally accepted wisdom on the connection between the weak localization and the strong localization is that enhanced backscattering of a diffusive wave packet leads to an effective reduction in the diffusion constant of the wave packet. When the influence of the increased backscattering is so overwhelming that the diffusion constant vanishes, the strong localization sets in.

It is worthy to mention that tremendous efforts have been devoted to investigate the localization phenomenon for classical waves in random media over the past several years [14–23]. However, observation of classical wave localization is a difficult task, partially because suitable systems are hard to find and partially because observation is often complicated by such effects as absorption and attenuation.

One important quantity associated with the wave localization is the localization length, which is defined as the characteristic decay length of the transmitted intensity through the system from the source. Taking into account the enhanced backscattering, the corrected (renormalized) diffusion constant may be obtained which then may be set to zero to obtain the localization length. This theoretical version of the localization length may be found in the literature for various dimensions for both the electronic waves and classical waves

(e.g., Refs. [14,24], and summarized in Ref. [25]).

A question we raise in this paper is about the validity of the existing theoretical formula for localization length, and subsequently the appropriateness or accuracy of the previous theory of wave localization. A main reason behind raising this question is that it would be helpful to experimentalists in designing experiments, provided that the theoretical formula for localization length is accurate enough in both qualitative and quantitative sense, and otherwise, it may mislead the experimentalists if the theory is not accurate enough.

The main aim of this paper is to answer the questions raised above for classical waves. One possible way is to concentrate on an exactly computable system and find the exact characteristic decay length of the transmitted intensity with the sample size, and compare with the theoretical value of this particular system, obtained from the existing theory. In this paper, we first outline the previous theory and show a numerically exact procedure for the calculation of the localization length for classical waves in two-dimensional media. Then we compare the two results, to gain information about the accuracy and appropriateness of the previous theory. In addition, we will also investigate the localization behavior as the frequency varies. These will be followed by a detailed discussion on the implications of the results.

II. FORMALISM

A. The theory

Here we briefly review the existing theory for wave localization. As wave propagates in random media, it experiences multiple random scattering, and as a result, the wave loses its propagating phase, leading to the gradual decreases of the coherence of the wave in the absence of absorption, i.e., elastic scattering. Meanwhile, diffusive wave is built up as more and more scattering takes place. The traditional picture of localization is in fact a version of localization of diffusive waves. In other words, the conventional wisdom towards the localization mechanism is the absence of diffusion. The procedure to obtain the localization state can be briefly summarized as follows.

The quantity $D^{(B)}$, which is a measure of diffusion of classical waves, is called the classical Boltzmann diffusion constant and it may be derived under the coherent potential

approximation [2], and is given as [25]

$$D^{(B)} \sim \frac{v_t l}{d_{dim}}, \qquad (1)$$

where v_t is the transport velocity, l is the mean free path, and d_{dim} is the dimensionality.

Waves scattered along any two reversed paths in the backward direction interfere constructively, leading to the enhanced backscattering effect, which will then add corrections to the diffusion coefficient. In the two-dimension case, such an enhanced backscattering effect is represented by a set of maximally crossed ladder diagrams. An evaluation of these diagrams leads to an integration for which two cutoff limits have to be introduced to avoid the divergence. In this way, the correction to the diffusion constant for two-dimensional system may be found to be [25]

$$\delta D \sim -\ln(L_M/l_m), \qquad (2)$$

where L_M and l_m are the two cutoff limits. It is then interpreted in the previous theory that the cutoff limit l_m is a measure of the minimum scaling for the waves and is thought to be related to (for example) the mean free path, whereas L_M is a measure of the effective size of the sample. It is rather important to note that the correction in Eq. (2) is not only negative but diverges as L_M approach infinity. This is obviously unphysical, since the conductance or the corrected diffusion constant cannot be negative. To avoid the problem, it was conjectured that L_M is in fact related to the localization range, or simply the localization length, in such way that when L_M is equal to the localization length denoted by ξ, say (i.e., $L_M = \xi$), the corrected diffusion coefficient becomes zero, i.e., the absence of diffusion:

$$D_R(\xi) = D^{(B)} + \delta D(\xi) = 0. \qquad (3)$$

The localization length ξ is solved from this equation. It is obvious that this equation always renders a solution and thus a localization length can always be found in two dimensions. Such an absence-of-diffusion mechanism is the core of the previous theory of localization in two dimensions, and supports the assertion from a scaling analysis that all waves are localized in two dimensions [26].

As will be shown later, there are significant discrepancies between the result from the absence-of-diffusion mechanism and the result from the exact numerical computation. These discrepancies would indicate either that the theory based on the diffusion mechanism is not accurate enough or simply that the theory has not yet fully captured the essence of the localization. We will come back to this later. Since the theory heavily depends on the diffusion picture, we may call it the *diffusion based theory*.

Equation (3) leads to the following solution for the localization length in two dimensions:

$$\xi_{\text{theory}} = l \exp\left[\frac{\pi}{2}\text{Re}(k_{\text{eff}})l\right]. \qquad (4)$$

Under the effective medium theory, the mean free path l may be obtained from the characteristic decay length of the coherent intensity and may be expressed as $l = 1/2 \, \text{Im}(k_{\text{eff}})$ and the effective wave number k_{eff} may be written as

$$k_{\text{eff}} = k + \sqrt{\frac{2\pi}{ik}\rho f(0)}, \qquad (5)$$

where k is the wave number of the incident wave propagating through the random scattering media with the scatterer numerical density ρ; $f(0)$ represents the scattering function of a single scatterer in the forward direction. The forward scattering function can be calculated by the standard series expansion method for a cylinder with radius a, when a wave with wave vector k is incident normally on it. The forward scattering function is given as

$$f(0) = \sqrt{\frac{2}{\pi k}}\sum_{n=0}^{\infty} C_n e^{-i(n\pi/2 + \pi/4)} \qquad (6)$$

with

$$C_n = \frac{-\epsilon_n\left[J_n'(ka)J_n(k'a) - \dfrac{1}{gh}J_n(ka)J_n'(k'a)\right]}{H_n^{(1)\prime}(ka)J_n(k'a) - \dfrac{1}{gh}H_n^{(1)}(ka)J_n'(k'a)},$$

where $\epsilon_0 = 1$ and $\epsilon_{n\geq 1} = 2$, g is the ratio of the density of the scatterer to that of the media, h is the ratio of the wave speed in the sactterer to that in the media, J_n represents the Bessel function of order n, $H_n^{(1)}$ represents the nth order Hankel function of first kind, and prime indicates the derivative.

B. Exact numerical calculation

Thus, on one hand the localization length of classical wave in a system consisting of any kind of cylindrical scatterers placed randomly in any media may be evaluated by the use of equations from the previous theory. On the other hand, the localization length of classical waves in two-dimensional random media may also be calculated exactly by the use of the multiple scattering technique.

Consider N straight cylinders located at $\vec{r}_i$ with $i = 1, 2, \ldots, N$ to form a *completely* random array. An acoustic line source transmitting monochromatic waves is placed at $\vec{r}_s$. Here we take the standard approach with regard to the source. That is, the transmission from the source is calculated from the multiple scattering theory, and assume that the source is not affected by the surroundings. If some other sources such as a line of atoms are used, the reaction between the source and the backscattered waves should take into account.

The scattered wave from each cylinder is a response to the total incident wave composed of the direct wave from the source and the multiply scattered waves from other cylinders. The final wave reaching a receiver located at $\vec{r}_r$ is the sum of direct wave from the source and the scattered waves from all the cylinders. Such scattering problem can be solved

exactly, following Twersky. While the details are in Ref. [27], the essential procedures are summarized below. The formulation presented below has been successfully applied to explain the recent experimental observations of acoustic crystals [28,29].

The scattered wave from the jth cylinder can be written as

$$p_s(\vec{r}, \vec{r}_j) = \sum_{n=-\infty}^{\infty} i\pi A_n^j H_n^{(1)}(k|\vec{r} - \vec{r}_j|)e^{in\phi_{\vec{r}-\vec{r}_j}}, \qquad (7)$$

where k is the wave number in the medium, $H_n^{(1)}$ is the nth order Hankel function of first kind, and $\phi_{\vec{r}-\vec{r}_j}$ is the azimuthal angle of the vector $\vec{r} - \vec{r}_j$ relative to the positive x axis. The total incident wave around the ith cylinder ($i = 1, 2, \ldots, N; i \neq j$) is summation of the direct incident wave from the source and the scattered waves from all other scatterers, can be expressed as

$$p_{in}^i(\vec{r}) = \sum_{n=-\infty}^{\infty} B_n^i J_n(k|\vec{r} - \vec{r}_i|)e^{in\phi_{\vec{r}-\vec{r}_i}}. \qquad (8)$$

The coefficients A_n^i and B_n^i can be solved by expressing the scattered wave $p_s(\vec{r}, \vec{r}_j)$, for each $j \neq i$, in terms of the modes with respect to the ith scatterer by the addition theorem for Bessel function [30]. Then the usual boundary conditions are matched at each individual scattering cylinder. This leads to

$$B_n^i = S_n^i + \sum_{j=1, j \neq i}^{N} C_n^{j,i}, \qquad (9)$$

with

$$S_n^i = i\pi H_{-n}^{(1)}(k|\vec{r}_i|)e^{-in\phi_{\vec{r}_i}}, \qquad (10)$$

$$C_n^{j,i} = \sum_{l=-\infty}^{\infty} i\pi A_l^j H_{l-n}^{(1)}(k|\vec{r}_i - \vec{r}_j|)e^{i(l-n)\phi_{\vec{r}_i-\vec{r}_j}}, \qquad (11)$$

and

$$B_n^i = i\pi \tau_n^i A_n^i, \qquad (12)$$

where τ_n^i are the transfer matrices relating the acoustic properties of the scatterers and the surrounding medium, and have been given by Eq. (21) in Ref. [28].

The coefficients A_n^i and B_n^j can then be determined from Eq. (9). Once the coefficients A_n^i are determined, the transmitted wave at any special point is given by

$$p(\vec{r}) = p_0(\vec{r}) + \sum_{i=1}^{N} \sum_{n=-\infty}^{\infty} i\pi A_n^i H_n^{(1)}(k|\vec{r} - \vec{r}_i|)e^{in\phi_{\vec{r}-\vec{r}_i}}, \qquad (13)$$

where p_0 is the field when no scatterers are present. The normalized transmission is defined as $T = p/p_0$ and, therefore, the acoustic intensity is represented as $|T|^2$.

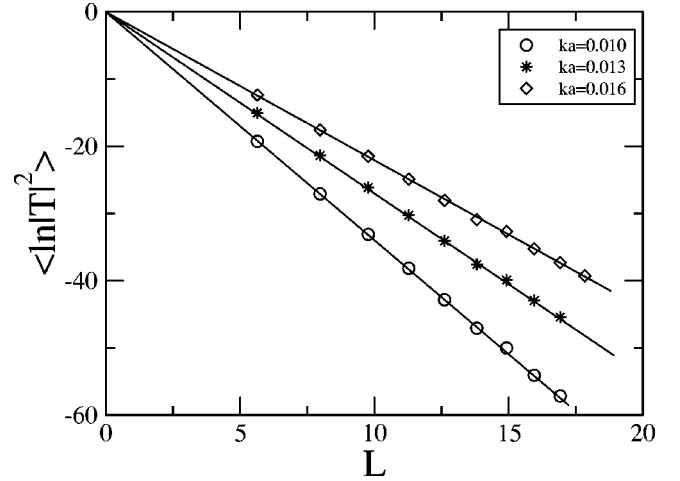The averaged localization length is subsequently determined by [31,32]

FIG. 1. $\langle \ln|T|^2 \rangle$ is plotted as a function of sample size L for $ka = 0.010$ (circles), $ka = 0.013$ (stars) and $ka = 0.016$ (diamonds), respectively. Here the filling factor β is 0.001.

$$\xi_{\text{exact}} = -\frac{L}{\langle \ln|T(L)|^2 \rangle}, \qquad (14)$$

where L is the sample size, and $\langle \cdot \rangle$ denotes the ensemble average over the random distribution of the scatterers. Thus obtained localization length can be compared to that in Eq. (4) obtained analytically from the previous theory.

III. RESULTS COMPARISON AND DISCUSSION

A. Results comparison

The system we consider here consists of N identical air cylinders placed randomly in water medium. The reason behind considering the system as the air cylinders in water is that due to the large contrast of densities for air and water, and also a large contrast of sound speeds in air and water, the air cylinders act as strong scatters to the waves propagating in water media. The radius of each air cylinder is a. The fraction of area occupied by the cylinders per area is β. The average distance between nearest neighbors is, therefore, $d = (\pi/\beta)^{1/2}a$, which is also the lattice constant for the corresponding regular lattice array. All the cylinders are placed completely randomly within a circle of radius L. A transmitting line source is located at the center and the receiver is located outside the scattering cloud. In the computation, the acoustic intensity is normalized in such a way that its value equals unity, when there are no scatterers present. Thus the uninteresting geometrical spreading effect is naturally eliminated. All the lengths are scaled by the parameter d, and the frequency is presented in terms of ka to make the computation nondimensional.

Figure 1 shows the variation of $\langle \ln|T|^2 \rangle$ as a function of the system size L for three different values of frequency ($ka = 0.01$, 0.013, and 0.016) with the filling fraction $\beta = 0.001$. The $\langle \cdot \rangle$ implies the configuration average of the total transmitted intensity. Number of configurations considered here is 200. It is apparent from the Fig. 1 that the averaged total transmitted intensity decays exponentially with the
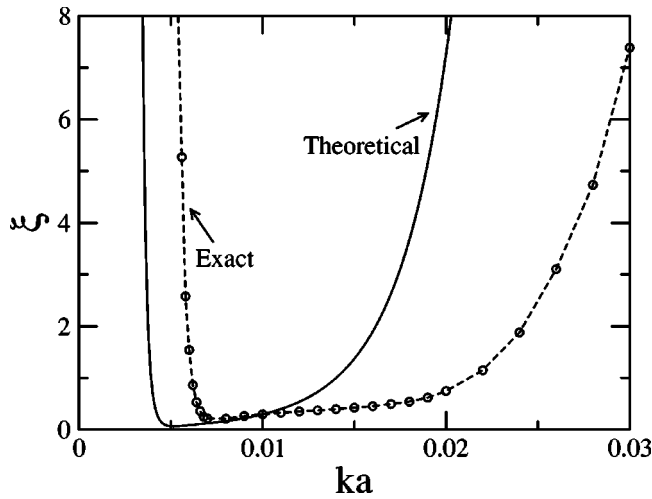
FIG. 2. Localization length ξ is shown as a function of frequencies ka for $\beta=0.001$. The dashed curve with circles represent the exact values obtained numerically, while the solid curve is obtained from theoretical formula.

FIG. 3. Same as the Fig. 2 except that $\beta=0.01$ here.

dimensionless system size, L for all the ka values shown in the figure, which in turn indicates the localization of wave at those frequencies. The localization length is nothing but the characteristic decay length and thus it may be obtained from the inverse of the slope of the straight lines (in Fig. 1) for those frequencies. Similarly one may obtain the localization length of classical waves at other frequencies from the exact numerical computation.

Figure 2 presents the dimensionless localization length ξ as a function of frequencies ka for $\beta=0.001$. The dashed curve with circles represent the localization length obtained from the exact numerical computation using the multiple scattering technique, while the solid curve represents the localization length calculated from the theoretical formula as prescribed earlier in Eq. (4). From the dashed curve, we observe that the localization length increases very rapidly as the frequency of the wave is decreased at the low frequency regime, the localization length also increases rapidly with the increase of frequency at the high-frequency regime and there is a minimum of localization length at an intermediate frequency. For lower values of frequencies, the waves propagating through the random media feel the media as homogeneous over a large distance depending on the wavelength, for higher values of frequencies the waves are like very localized objects that can easily propagate between the scatterers and at an optimal frequency, the waves feel the strongest scattering in the random media. Thus, the larger values of localization lengths at lower and higher frequencies and the minimum value of localization length at an intermediate frequency are understood physically.

However, we observe that the critical change in the localization length takes place at lower and higher frequencies. On the other hand, the solid curve also shows similar behavior. But, one important difference is that the minimum of the dashed curve occurs at a higher frequency when compared with that of the solid curve. In other wards, the strongest localization predicted from the previous theoretical formula occurs at a frequency, which is lower than that obtained from
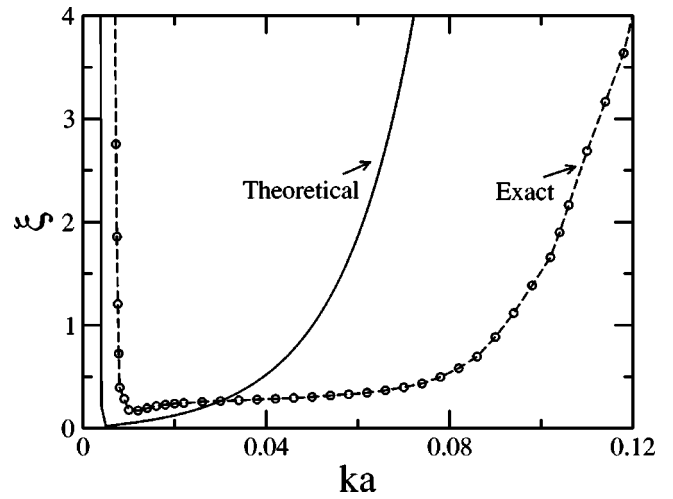
the exact numerical computation. As a result, we observe that the two curves cross each other at some frequency, say $(ka)_{cr}$. For $ka<(ka)_{cr}$, the localization length obtained from exact numerical calculation is larger than that obtained from the existing theoretical formula while for $ka>(ka)_{cr}$, the theoretical value dominates over the exact value of localization length. [Note that both the results are theoretical though, the results (values) obtained from the previous theoretical formula are termed as theoretical results (values) and those obtained from exact numerical calculations are termed as exact results (values) and we may use this convention for our convenience.] In the low- and high-frequency regimes, the discrepancies between the theoretical and the exact values are significant.

We have also done similar calculations for $\beta=0.01$ and compared the localization length obtained from the theoretical formula with that obtained from the exact numerical computation and is shown in Fig. 3. While the theoretical formula seems to capture the shape of the exact results (Figs. 2 and 3), but there are significant discrepancies with regard to whether the waves are most localized, and where localization length starts to increase rapidly.

We have shown in Fig. 4, the variation of localization length ξ as a function of the filling factor β for $ka=0.009$, which is less than $(ka)_{cr}$ in Fig. 2. The dashed curve with circle represents the exact localization length obtained from numerical calculation, and the solid curve is obtained from theoretical formula. The solid curve shows the monotonous decrease of the localization length as the filing factor increases but the exact result (dashed curve with circle) shows completely different behavior: the localization length initially decreases and then increases monotonically, leaving a minimum at some value of β. Thus, the wave with frequency $ka=0.009$ becomes localized most strongly only for a specific value of the average density of scatterers. However, for $ka=0.009$, the exact localization length is larger than the theoretical values for all filling factors. Thus, here we observe that at the frequency $ka=0.009$, the theoretical results do not capture the overall trend of the exact results. The eventual increase of the localization length with increasing β
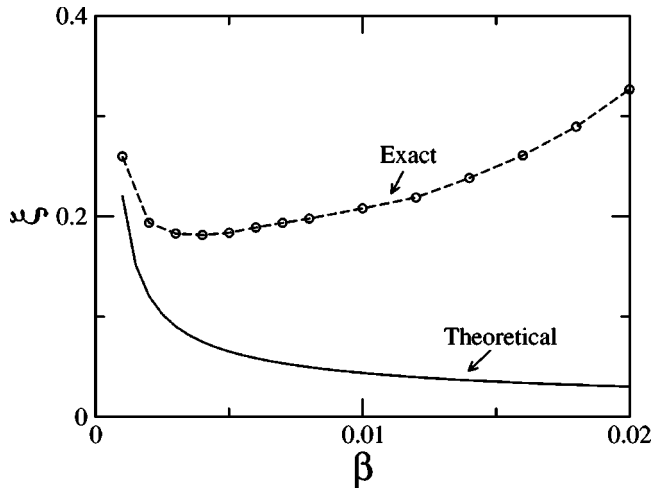
FIG. 4. Localization length ξ is shown as a function of filling factor β for $ka=0.009$. The dashed curve with circles represent the exact values obtained numerically, while the solid curve is obtained from theoretical formula.

is understandable. At two limits, i.e., for either pure water ($\beta=0$) or pure air ($\beta=1$), there can be no localization and the localization length must approach infinity. Therefore, the curve of the localization length versus β has to be in the shape of valley as shown in Fig. 4. The theoretical result is based upon the approximation that air cylinders are perturbative scatterers in the medium of water, thus the theory is expected to fail for strong scattering and thus cannot work for large β. This would explain, why the theory cannot capture the overall trend of the localization length as a function of β.

The variation of localization length ξ as a function of the filling factor β for a moderate frequency $ka=0.015$, which is greater than $(ka)_{cr}$ in Fig. 2, is shown in Fig. 5. The dashed curve with circle represents the exact result and the solid curve represents the theoretical result for localization length. The localization length decreases with filling factor for both the curves. However, there is a cross over between two curves at some value of β, say β_{cr} below which the theoretical value is higher than the exact value and above β_{cr}
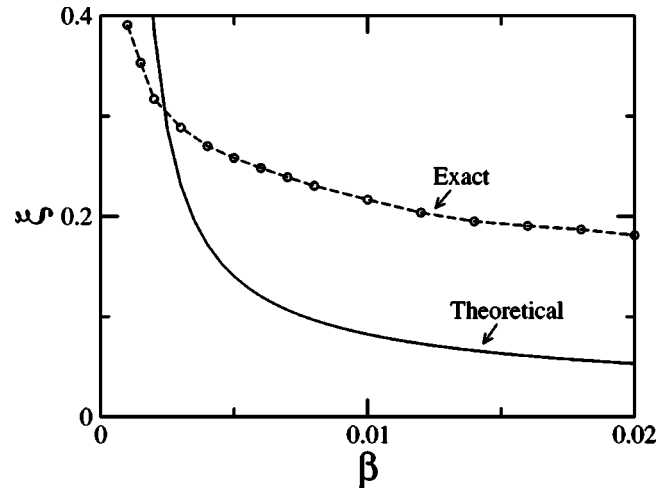

FIG. 5. Same as the Fig. 4 except that $ka=0.015$ here.

the exact values are larger than the theoretical values for localization length. Thus at $ka=0.015$, the exact and theoretical curves are similar in shape for the range of β considered, the exact and theoretical values for localization length matches at $\beta=\beta_{cr}$, otherwise there are significant quantitative discrepancies. We note that when increasing β further, like in Fig. 4, the valley shape for the exact result will appear; the explanation can be referred to from the above discussion. All the observed discrepancies between the theoretical and exact results are probably due to the approximations involved in the previous theory.

In order to study the fluctuation behavior of localization, we define an exponent, by analogy with one-dimensional cases [33],

$$\gamma \equiv -\frac{1}{L}\ln(|T|^2), \qquad (15)$$

which is reciprocal to the localization length. Figure 6(a) shows the variance of γ, i.e., $var(\gamma)=\langle\gamma^2\rangle-\langle\gamma\rangle^2$, as a function of ka for $\beta=0.001$.

We interestingly observe that for a range of frequencies, the fluctuation is rather small indicating that the wave is
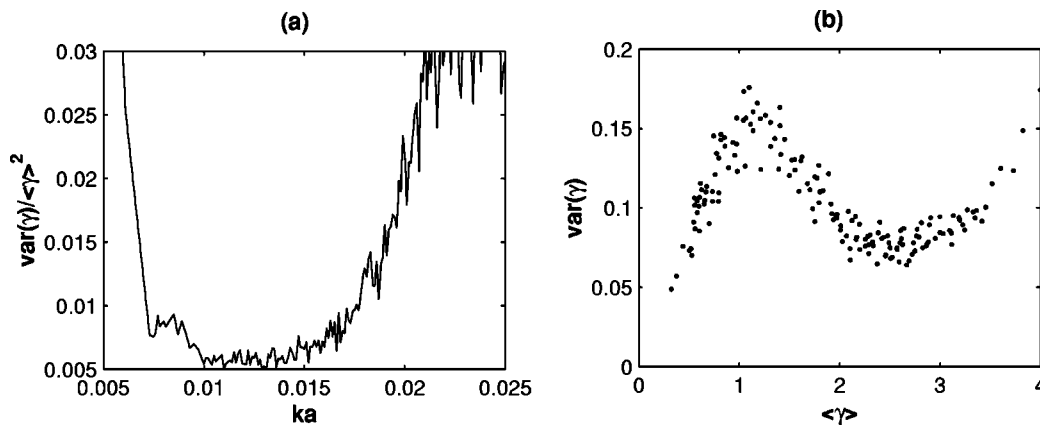

FIG. 6. (a) This shows the variance of the localization exponent $var(\gamma)$ as a function of frequency ka with $\beta=0.001$. (b) The variance $var(\gamma)$ is shown as a function of the mean $\langle\gamma\rangle$ for $\beta=0.001$.
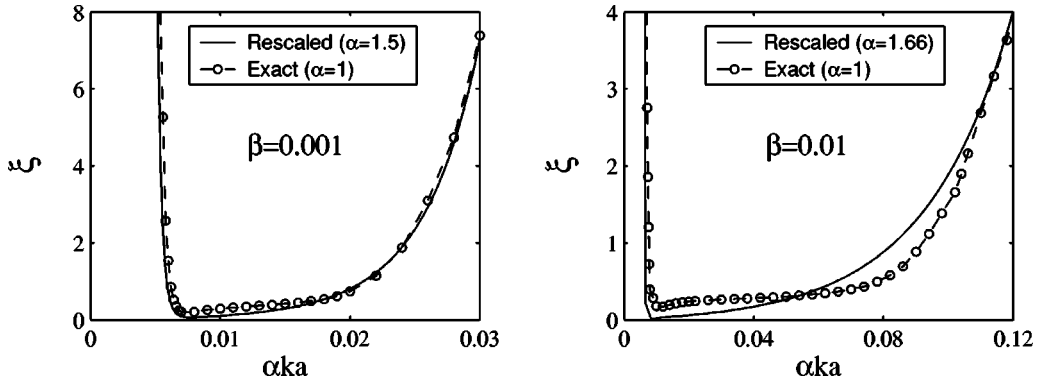
FIG. 7. Scaled replots of Figs. 2 and 3. In the plots, α denotes the scaling factor. The theoretical results are plotted vs αka for two α values.

strongly localized in this range of frequencies. However, at low and high frequencies (below and above the range of frequencies where the fluctuation is very small) the fluctuation is high. This is an indication of some critical change in the localization behavior and it tends to suggest that waves become very rapidly and critically less localized at low and high frequencies. It may also be an implication of the localization-delocalization transition, as discussed in Refs. [34,35]. These cannot be explained in the context of the current diffusion based theory. We also plot the variance of the exponent γ versus the mean $\langle \gamma \rangle$ in Fig. 6(b). Here we do not observe the linear dependence between the mean localization length and the variance, similar to one-dimensional cases [33,36].

B. Discussion

All the deviations from the predictions of the existing theory should possibly have a couple of important implications. The first is merely that the theory is not accurate enough, since it is necessarily based upon a perturbation scheme given the complications involved in the problem. If this scenario is valid, we could believe that though inaccurate, the present theory still captures the fundamental nature of the localization phenomenon. It would then be expectable that the discrepancies will be reduced when more and more relevant multiple scattering processes are included in the theoretical derivation. A next task would thus be to look for these scattering processes.

In fact, by inspection, it is evident that the exact and theoretical results in Figs. 2 and 3 seem to be able to match each other by a scaling. In Fig. 7, we replot the results from Figs. 2 and 3 by rescaling the variable ka for the exact results. The scaling factor α is 1.5 and 1.66 for $\beta = 0.001$ and 0.01, respectively. Here we see that after the scaling, the two results match quite well, particularly for low β. Such scaling is equivalent to replacing k in Eq. (5) by k/α, suggesting a need for renormalizing the parameters used in the theory, possibly those that come from the effective medium approximation, as suggested by one of the referees. A few notes can be made on the scaling factor.

(1) As shown in Fig. 7, the scaling factor depends on β, and appears to increase with β.

(2) The scaled match degrades as β increases.

(3) Looking again at Figs. 4 and 5, we can infer that for frequencies near the strongest localization, i.e., strongest scattering, the scaled match is expected to degrade faster than other frequencies as β increases; for example, comparing Fig. 4 to Fig. 5, we can see that the exact and theoretical results for $ka = 0.015$ have the similar shapes for a wider range of β.

All these would imply that the scaled match is better for weak scattering, and the level of match decreases as the scattering increases which can be made by adding more scatterers per unit area or moving to strong scattering regions. It can also be inferred that Eq. (4) may capture the localization essence, but there is a need for improvement of the effective medium theory that leads to Eq. (5).

The second implication would be simply that the existing theory, which is based upon the absence-of-diffusion mechanism, has not yet fully encapsulated the essence of the phenomenon of wave localization. Beside the results shown above, there are other evidences indicating that this line of reasoning should not be discarded too lightly. For example, it has been shown that in the localized state, a genuine phase-coherence behavior prevails as a unique feature of localized waves [34]. This coherent behavior can be understood as follows. For quantum mechanic or acoustic waves, the current can be written as $\vec{J} \sim \mathrm{Re}(\psi(-i)\nabla\psi)$, where ψ stands for the wave function for quantum mechanical systems and for the pressure in acoustic systems. Writing the field as $\psi = |\psi|e^{i\theta}$, the current becomes $\vec{J} \sim |\psi|^2 \nabla\theta$. It is clear that when θ is constant at least by domains while $|\psi| \neq 0$, the flow stops, i.e., $\vec{J} = 0$, and the wave or the energy is localized in space, i.e., $|\psi|^2 \neq 0$. Obviously the constant phase θ indicates the appearance of a long range ordering in the system. All these have been demonstrated successfully not only for two-dimensional media [34,35], but for one and three dimensions as well [21,36,37]. It can be easily shown that the same consideration also holds for electromagnetic waves [38]. In the current diffusion based theory, the phase information is not attained. Therefore, at least the theory cannot account for the phase coherence related localization phenomenon. Additionally, the theory could also be expected to fail when waves come to a complete stop before the diffusion becomes

dominant. In summary, if the above arguments are valid, the next question would be whether these can be amended by improving the effective medium theory.

IV. SUMMARY

The localization length of classical waves in two-dimensional random media is obtained from exact numerical computation as well as from the previous theoretical formula and then the results are compared. Significant discrepancies between the two results is observed. Furthermore, results indicate some critical change in the localization behavior at low and high frequencies, and it seems to suggest that waves become very rapidly and critically less localized at these fre-

quencies. These features are not readily seen in the previous theoretical picture of wave localization. However, by a rescaling of parameters the two results tend to match each other for weak scattering. Therefore, it is suggested that caution should be taken when applying the previous analytic theory to infer the localization length of classical waves. Possible implications of the present research are discussed.

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